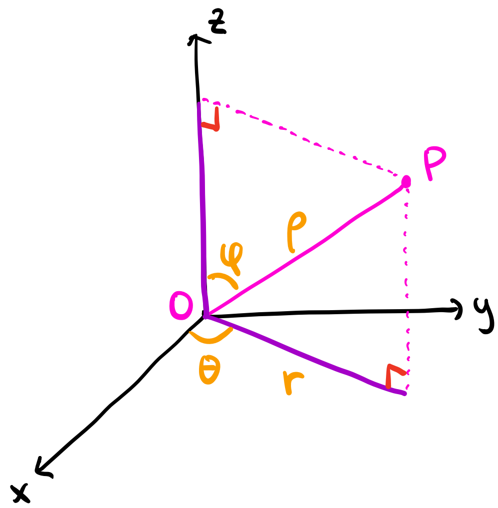
15.7 + 15.8. Triple integrals in cylindrical / spherical coordinates

Def (1) Cylindrical coordinates are given by polar coordinates together with z-coordinates.

$\Rightarrow x = r\cos\theta,\ y = r\sin\theta,\ z = z$

(2) Spherical coordinates are related to rectangular coordinates by $x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$.



ρ: the distance from $O = (0,0,0)$

$\Rightarrow \rho = \sqrt{x^2+y^2+z^2}$

θ: the angle parameter in polar coordinates

φ: the angle between OP and the positive z-axis.

Note (1) The two coordinate systems are related by

$r = \rho\sin\varphi,\ \theta = \theta,\ z = \rho\cos\varphi$.

(2) φ only takes value from 0 to π.

($\because \pi < \varphi < 2\pi \Rightarrow \sin\varphi < 0 \Rightarrow r = \rho\sin\varphi < 0$)

Prop If $f(x,y,z)$ is a continuous function on a solid E,

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\,\underline{r}\,dz\,dr\,d\theta$$

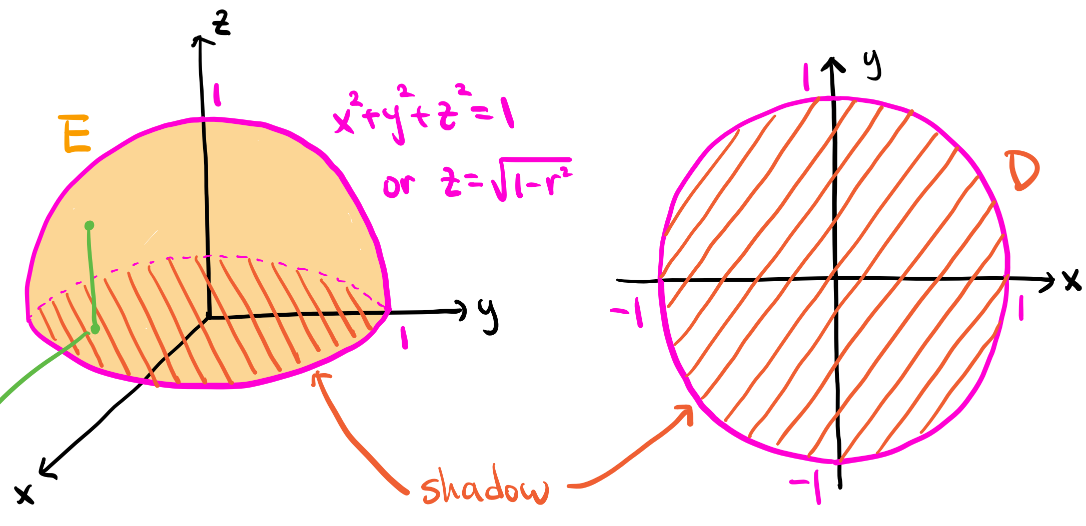
Jacobian

$$= \iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\underline{\rho^2\sin\varphi}\,d\rho\,d\varphi\,d\theta$$

Ex Let E be the solid given by $x^2+y^2+z^2 \le 1$ and $z \ge 0$.

(1) Describe the bounds for E in cylindrical coordinates.

Sol



$x^2+y^2+z^2=1 \rightsquigarrow r^2+z^2=1 \rightsquigarrow z=\sqrt{1-r^2}$

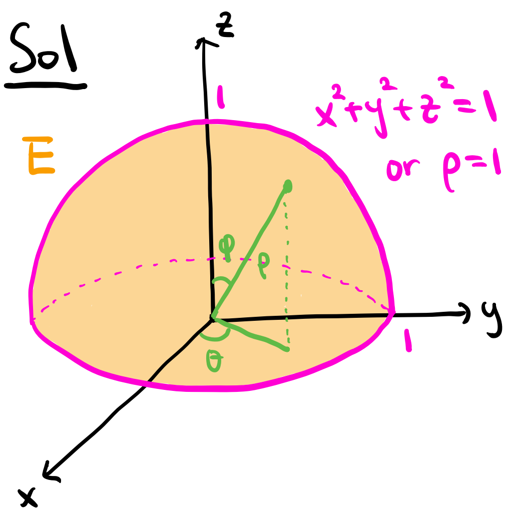
The shadow D on the xy-plane: $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

For each point on D: $0 \le z \le \sqrt{1-r^2}$.

$\Rightarrow \boxed{0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad 0 \le z \le \sqrt{1-r^2}}$

(2) Describe the bounds for E in spherical coordinates.

Sol



$x^2+y^2+z^2=1 \rightsquigarrow \rho^2=1 \rightsquigarrow \rho=1$.

$z=0 \rightsquigarrow \rho\cos\varphi=0 \rightsquigarrow \cos\varphi=0$

$\rightsquigarrow \varphi=\frac{\pi}{2}$.

$0 \le \varphi \le \pi$.

$\Rightarrow \boxed{0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \frac{\pi}{2}, \quad 0 \le \rho \le 1}$

(3) Find the center of mass with density $\rho(x,y,z)=1$.

<u>Sol</u> $m = \iiint_E \rho(x,y,z)\,dV = \iiint_E 1\,dV$

$$= \text{vol}(E) = \frac{1}{2}\cdot \underbrace{\frac{4}{3}\pi\cdot 1^3}_{\text{volume of sphere}} = \frac{2\pi}{3}$$

E is symmetric about the yz, xz planes.

$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\,dV = \frac{1}{m}\iiint_E \underbrace{x}_{\text{odd w.r.t } x}\,dV = 0$$

$$\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\,dV = \frac{1}{m}\iiint_E \underbrace{y}_{\text{odd w.r.t } y}\,dV = 0$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV = \frac{1}{m}\iiint_E z\,dV$$

Jacobian

$$= \frac{3}{2\pi}\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} z\cdot r\,dz\,dr\,d\theta$$

$$= \frac{3}{2\pi}\int_0^{2\pi}\int_0^1 \frac{z^2}{2}\cdot r\Big|_{z=0}^{z=\sqrt{1-r^2}}\,dr\,d\theta$$

$$= \frac{3}{2\pi}\int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta = \frac{3}{2\pi}\int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big|_{r=0}^{r=1}\,d\theta$$

$$= \frac{3}{2\pi}\int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{3}{2\pi}\cdot 2\pi\cdot\frac{1}{4} = \frac{3}{8}$$

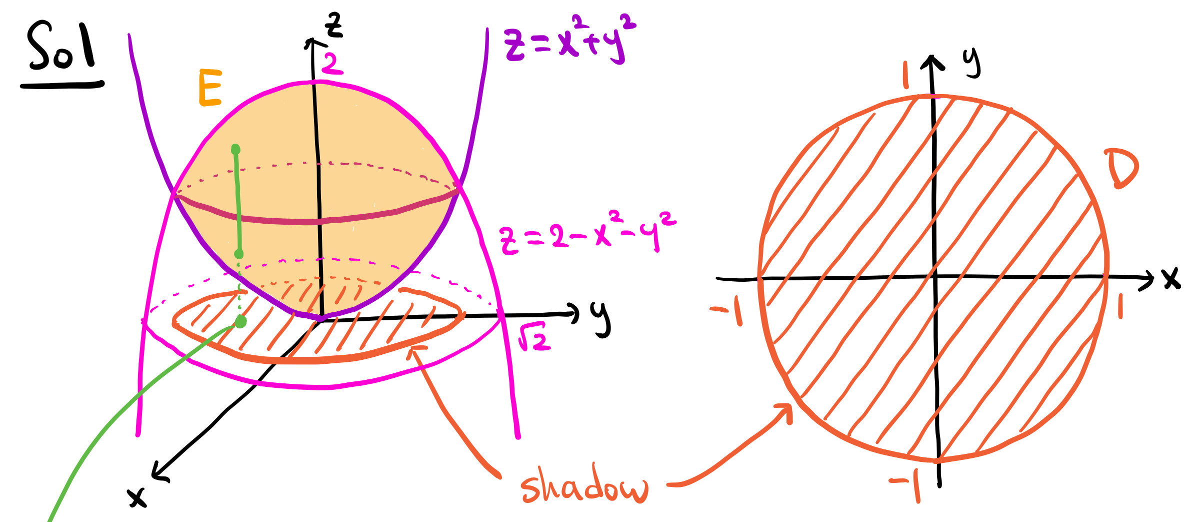
$\Rightarrow$ The center of mass is $\boxed{(0,0,\frac{3}{8})}$

Jacobian

<u>Note</u> You also have $\bar{z} = \dfrac{3}{2\pi}\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos\varphi\cdot \underline{\rho^2\sin\varphi}\,d\rho\,d\varphi\,d\theta$

Ex Find the volume of the solid bounded by the paraboloids $z = x^2+y^2$ and $z = 2-x^2-y^2$.

Sol



In cylindrical coordinates:

$z = x^2+y^2 \rightsquigarrow z = r^2, \quad z = 2-x^2-y^2 \rightsquigarrow z = 2-r^2.$

Intersection : $z = r^2$ and $z = 2-r^2$

$\Rightarrow r^2 = 2-r^2 \Rightarrow r^2 = 1 \Rightarrow r = 1.$

The shadow on the xy-plane: $0 \le \theta \le 2\pi, \quad 0 \le r \le 1.$

For each point on the shadow: $r^2 \le z \le 2-r^2.$

$\Rightarrow 0 \le \theta \le 2\pi, \ 0 \le r \le 1, \ r^2 \le z \le 2-r^2.$

Jacobian

$$\mathrm{Vol}(E) = \iiint_E 1\, dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} 1 \cdot r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2-2r^2)\cdot r \, dr\, d\theta = \int_0^{2\pi}\int_0^1 2r - 2r^3 \, dr\, d\theta$$

$$= \int_0^{2\pi} r^2 - \frac{r^4}{2}\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta$$

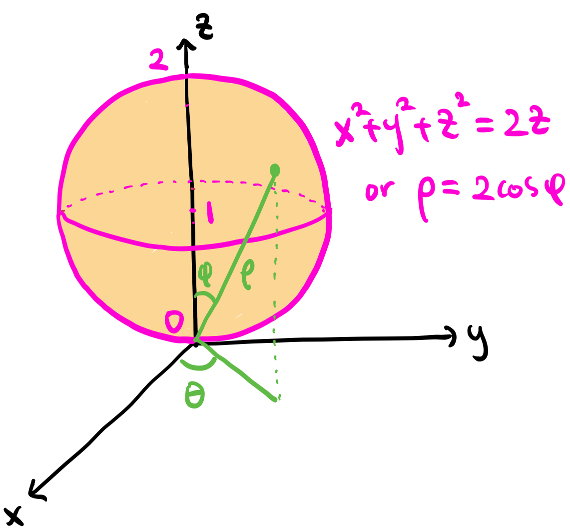
$$= 2\pi \cdot \frac{1}{2} = \boxed{\pi}$$

Ex Let E be the solid given by $x^2+y^2+z^2 \le 2z$.

Find the mass of E with density $\rho(x,y,z) = z$.

Sol $x^2+y^2+z^2 = 2z \rightsquigarrow x^2+y^2+z^2-2z+1 = 1 \rightsquigarrow x^2+y^2+(z-1)^2 = 1$

$\rightsquigarrow$ a sphere of radius 1 and center $(0,0,1)$.



In spherical coordinates:

$x^2+y^2+z^2 = 2z \rightsquigarrow \rho^2 = 2\rho\cos\varphi$

$\rightsquigarrow \rho = 2\cos\varphi$.

φ is maximized on the xy-plane.

$z=0 \rightsquigarrow \rho\cos\varphi = 0 \rightsquigarrow \cos\varphi = 0$

$\rightsquigarrow \varphi = \frac{\pi}{2}$ $(\because 0 \le \varphi \le \pi)$

$\Rightarrow 0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \frac{\pi}{2}, \quad 0 \le \rho \le 2\cos\varphi.$

$$m = \iiint_E \rho(x,y,z)\,dV = \iiint_E z\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho\cos\varphi \cdot \underbrace{\rho^2\sin\varphi}_{\text{Jacobian}}\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^4}{4}\cos\varphi\sin\varphi \Big|_{\rho=0}^{\rho=2\cos\varphi} d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 4\cos^5\varphi\sin\varphi\,d\varphi\,d\theta$$

$(u = \cos\varphi \Rightarrow du = -\sin\varphi\,d\varphi)$

$$= \int_0^{2\pi}\int_1^0 -4u^5\,du\,d\theta = \int_0^{2\pi} -\frac{2}{3}u^6 \Big|_{u=1}^{u=0} d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}\,d\theta = 2\pi \cdot \frac{2}{3}\,d\theta = \boxed{\frac{4\pi}{3}}$$